Natural convection boundary-layer adjacent to an inclined flat plate subject to sudden and ramp heating

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A B S T R A C T

The natural convection thermal boundary-layer adjacent to an inclined flat plate subject to sudden heating and a temperature boundary condition which follows a ramp function up until a specified time and then remains constant is investigated. The development of the flow from start-up to a steady state has been described based on scaling analyses and verified by numerical simulations. Different flow regimes based on the Rayleigh number are discussed with numerical results for both boundary conditions. For ramp heating, the boundary-layer flow depends on the comparison of the time at which the ramp heating is completed and the time at which the boundary layer completes its growth. If the ramp time is long compared with the steady-state time, the layer reaches a quasi-steady mode in which the growth of the layer is governed solely by the thermal balance between convection and conduction. On the other hand, if the ramp is completed before the layer becomes steady, the subsequent growth is governed by the balance between buoyancy and inertia, as for the case of instantaneous heating.

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1. Introduction

Natural convection is a very common phenomenon in nature. Natural convection along an inclined plate has received less attention than the classic cases of vertical and horizontal plates. However, natural convection heat transfer from an inclined surface is very frequently encountered in engineering devices and the natural environment. A large body of literature exists about an inclined semi-infinite flat plate because of its engineering application [1–4]. Most of the previous works have been conducted by either numerical simulations or experimental observations. Theoretical or scaling analyses have not played a significant role for this type of problem, especially with regard to the transient flow behavior from the start-up, which is of great fundamental interest and has practical importance. In contrast to the inclined plate problem, very detailed scaling analysis has been carried out for the transient flow in rectangular cavities with differentially heated sidewalls by Patterson and Imberger [5], and theoretical analyses of triangular cavities with a sloping bottom have also been reported in the context of natural convection induced circulation in coastal waters [6,7]. Scaling method, rules and applications are also taught in the text book [8].

Scale analysis is a cost-effective way that can be applied as a first step in understanding the physics underlying the fluid flow and heat transfer issues. The results of scale analysis can serve as a guide for both experimental and numerical investigations. Therefore, scaling has been used by many researchers to investigate the transient flow development for different kinds of geometries and thermal forcing. Patterson and Imberger [5] carried out an extensive investigation of the transient behavior of natural convection of a two-dimensional rectangular cavity in which the two opposing vertical sidewalls are simultaneously heated and cooled by an equal amount. The authors proposed several flow regimes of the transient flow development based on the relative values of the Rayleigh number $Ra$, the Prandtl number $Pr$, and the aspect ratio of the cavity $A$. Schladow et al. [9] conducted a series of two- and three-dimensional numerical simulations of the transient flow in a side-heated cavity, and their simulations generally agree with the results of the scaling arguments of Patterson and Imberger [5].

Scaling analyses coupled with numerical simulations have been used in a variety of other geometries and thermal forcing. For example, very recently, Lin and Armfield [10–12] investigated the transient processes of cooling an initially homogeneous fluid by natural convection in a vertical circular cylinder and in a rectangular container.

To identify possible flow regimes of the unsteady natural convection flow in a small-slope shallow wedge induced by the absorption of solar radiation, Lei and Patterson [6] presented a scaling
analysis and established relevant scales to quantify the flow properties in each flow regime. They classified the flow development broadly into one of three regimes: a conductive regime, a transitional regime and a convective regime, depending on the Rayleigh number.

Scaling analysis of the transient behavior of the flow in an attic space was conducted by Poulikakos and Bejan [13], valid for shallow spaces i.e. $H/B \to 0$, where $H$ and $B$ are the attic height and length respectively. The transient phenomenon began with the sudden cooling of the upper slopped wall. It was noted that both walls developed thermal and viscous layers whose thickness increased towards steady-state values. The authors mentioned that, by properly identifying the time scales of various features that develop inside the enclosures, it was possible to predict theoretically the basic flow features that would endure in the steady state. Finally, they focused on a complete sequence of transient numerical simulations covering a range of controlling parameters including the Rayleigh number, the aspect ratio and the Prandtl number.

A scaling analysis is presented by Patterson et al. [14] for the transient boundary layer established on a vertical wall for Prandtl number $Pr > 1$ following non-instantaneous heating in the form of an imposed wall temperature which increases linearly up to a prescribed steady value over a prescribed time. The authors also verify their scaling relations with numerical solutions of the full equations of motion and energy. They reveal many interesting time scales for the boundary-layer development. Recently scaling analysis for both sudden and ramp cooling boundary conditions has been performed for the inclined walls of an attic space by Saha et al. [15]. In addition to the boundary-layer analysis, the authors established time scales for the cooling down of the whole cavity for both boundary conditions.

In this study, the behavior of the two-dimensional transient natural convection flows adjacent to a sudden and ramp heated inclined flat plate is investigated by scaling analysis and numerical simulation for $Pr < 1$ by a heated flat plate. The physical system sketched in Fig. 1 consists of an inclined flat plate ($CD = L$). We extend both ends of the plate by a distance equal to its length and form a rectangular domain, which is filled with an initially stationary fluid at a temperature $T_h$. If we consider the plate as the hypotenuse of a right angled triangle then the altitude is $h$, the length of the base is $l$ and the angle that the plate makes with the base is $\theta$. Except for the plate (the section CD shown in Fig. 1), all walls of the rectangular

<table>
<thead>
<tr>
<th>Nomenclature</th>
<th>Greek symbols</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$ slope of the plate</td>
<td>$\alpha$ thermal diffusivity</td>
</tr>
<tr>
<td>$L$ length of the plate</td>
<td>$\beta$ thermal expansion coeff.</td>
</tr>
<tr>
<td>$l$ length of the horizontal projection of the plate</td>
<td>$\Delta T$ temperature difference between hot surface and the ambient</td>
</tr>
<tr>
<td>$h$ height of the plate</td>
<td>$\Delta t$ time step</td>
</tr>
<tr>
<td>$g$ acceleration due to gravity</td>
<td>$\delta_{t_0}$ thickness of the thermal boundary layer at quasi-steady time</td>
</tr>
<tr>
<td>$k$ thermal conductivity</td>
<td>$\delta_{t_1}$ thickness of the thermal boundary layer at quasi-steady stage</td>
</tr>
<tr>
<td>$P$ pressure</td>
<td>$\delta_{t_0}$ thickness of the thermal boundary layer when the ramp is finished</td>
</tr>
<tr>
<td>$Pr$ Prandtl number</td>
<td>$\delta_{v_0}$ thickness of the viscous boundary layer at quasi-steady time</td>
</tr>
<tr>
<td>$Ra$ Rayleigh number</td>
<td>$\delta_{v_1}$ thickness of the viscous boundary layer at quasi-steady stage</td>
</tr>
<tr>
<td>$T$ temperature</td>
<td>$\delta_{v_2}$ thickness of the viscous boundary layer at quasi-steady time</td>
</tr>
<tr>
<td>$t$ time</td>
<td>$\kappa$ thermal diffusivity</td>
</tr>
<tr>
<td>$t_s$ steady-state time</td>
<td>$\rho$ density</td>
</tr>
<tr>
<td>$t_{s_r}$ quasi-steady time</td>
<td>$\nu$ kinematic viscosity</td>
</tr>
<tr>
<td>$t_r$ ramp time</td>
<td>$\theta$ angle</td>
</tr>
<tr>
<td>$T_c$ cooling temperature</td>
<td></td>
</tr>
<tr>
<td>$T_h$ heating temperature</td>
<td></td>
</tr>
<tr>
<td>$u, v$ velocity components</td>
<td></td>
</tr>
<tr>
<td>$u_s$ steady-state velocity</td>
<td></td>
</tr>
<tr>
<td>$u_{s_r}$ quasi-steady velocity</td>
<td></td>
</tr>
<tr>
<td>$x, y$ coordinates</td>
<td></td>
</tr>
</tbody>
</table>

| $c$ cooling temperature       |                                |
| $h$ height of the plate       |                                |
| $l$ length of the plate       |                                |
| $L$ slope of the plate        |                                |
| $A$ length of the horizontal projection of the plate |                                |
| $h$ height of the plate       |                                |
| $g$ acceleration due to gravity |                                |
| $k$ thermal conductivity      |                                |
| $P$ pressure                  |                                |
| $Pr$ Prandtl number           |                                |
| $Ra$ Rayleigh number          |                                |
| $T$ temperature               |                                |
| $t$ time                      |                                |
| $t_s$ steady-state time        |                                |
| $t_{s_r}$ quasi-steady time    |                                |
| $t_r$ ramp time               |                                |
| $T_c$ cooling temperature     |                                |
| $T_h$ heating temperature     |                                |
| $u, v$ velocity components    |                                |
| $u_s$ steady-state velocity    |                                |
| $u_{s_r}$ quasi-steady velocity |                                |
| $x, y$ coordinates            |                                |

Under consideration is the flow behavior resulting from the heating of an initially motionless and isothermal Newtonian fluid with $Pr < 1$ by a heated flat plate. The physical system sketched in Fig. 1 consists of an inclined flat plate ($CD = L$). We extend both ends of the plate by a distance equal to its length and form a rectangular domain, which is filled with an initially stationary fluid at a temperature $T_h$. If we consider the plate as the hypotenuse of a right angled triangle then the altitude is $h$, the length of the base is $l$ and the angle that the plate makes with the base is $\theta$. Except for the plate (the section CD shown in Fig. 1), all walls of the rectangular

![Fig. 1. Schematic of the computational domain and boundary conditions.](image-url)
domain are assumed to be adiabatic, rigid and non-slip. Two heating boundary conditions are applied on the plate; sudden heating to a specified temperature which is then maintained, and heating by a linearly increasing temperature to a specified temperature over certain time (the ramp time) after which the temperature is maintained (the ramp function). The ramp function is described in the scaling section in this chapter below.

The development of natural convection adjacent to the inclined plate is governed by the following dimensionless two-dimensional Navier–Stokes and energy equation with the Boussinesq approximation:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]  
(1)

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + g\beta \sin \theta (T - T_c)
\]  
(2)

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + g\beta \cos \theta (T - T_c)
\]  
(3)

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k}{\rho c_p} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)
\]  
(4)

where \(u\) and \(v\) are the \(x\)-direction and \(y\)-direction velocity components, \(t\) the time, \(p\) the pressure, \(T\) the temperature, and \(\beta, u\) and \(\kappa\) are respectively the thermal expansion coefficient, kinematic viscosity and thermal diffusivity of the fluid at \(T_c\).

3. Scaling for sudden heating

With the initiation of the flow, a thermal boundary-layer develops adjacent to the heated inclined plate. The parameters characterizing the boundary-layer development are predominantly the thermal boundary-layer thickness \(\delta_T\), the maximum velocity parallel to the plate \(u_w\) within the boundary layer, and the time \(t_s\) for the boundary layer to reach steady state.

3.1. Growth of the thermal boundary layer

As mentioned above, the sudden heating of the flat plate results in a thermal boundary layer developing adjacent to the inclined plate. We follow the arguments given by Patterson and Imberger [5], appropriately modified for the inclined plate and the Prandtl number (\(\text{Pr} < 1\)).

The energy equation (3) indicates that since the fluid is initially motionless the heating effect of the plate will first diffuse into the fluid layer through pure conduction, resulting in a thermal boundary layer of thickness \(\delta_T\). Within the boundary layer, the dominant balance is between the unsteady and diffusion terms in the energy equation (3), that is,

\[
\frac{\Delta T}{t} \sim \kappa \frac{\Delta T}{\delta_T^2},
\]

which leads to a scale for the thickness of the thermal boundary layer

\[
\delta_T \sim \sqrt[1/2]{\frac{\kappa}{t}}.
\]  
(5)

This scaling is valid till the convection term becomes important. The unsteady inertia term of the momentum equation (2) is \(O(u_t)\), the viscous term \(O(\nu u/\delta_T^2)\), and the advection term \(O(u^2/L)\). The ratio of the advection term to the unsteady term is then \(O(u_t/L)\). For very small time \(ut/L \ll 1\). Therefore the advection term is not significant for small time. The ratio of the unsteady to viscous term is \((u_t)/(\nu u/\delta_T^2) \sim \delta_T^2/(\nu t) \sim 1/\text{Pr}\), where \(\text{Pr} = \nu/\kappa\). For \(\text{Pr} \ll 1\) the viscous term is much smaller than the unsteady term, and therefore, the correct balance is between the unsteady term and buoyancy. However, for \(\text{Pr} \gg 1\), the unsteady term is much smaller than the viscous term and the correct balance is between viscosity and buoyancy. If \(\text{Pr} \sim 1\), then the unsteady and viscous terms are of the same order, and thus both terms need to be included in a balance with the buoyancy term. This balance was introduced by Lin et al. [16].

The unsteady term is \(O(u/t)\) and the viscous term is \(O(\text{Pr} u/t)\), so these two terms together are \(O(1+\text{Pr} u/t)\). Now the balance in the inclined momentum equation is

\[
(1 + \text{Pr}) \frac{u}{T} \sim g\beta \Delta T \sin \theta.
\]  
(6)

Therefore \(u \sim g\beta \sin \theta \Delta T t/(1 + \text{Pr})\). The inclination angle, \(\theta\) is related to the slope or aspect ratio \(A\) through \(\sin \theta = A/(1 + A^2)^{1/2}\). Hence (6) becomes

\[
\frac{u}{T} \sim \frac{A \text{RaPr}}{(1 + \text{Pr})(1 + A^2)^{1/2}} \left( \frac{t}{h^2/\kappa} \right) \left( \frac{k}{h} \right).
\]  
(7)

where the Rayleigh number is defined as \(\text{Ra} = g\beta \Delta T h^3/\nu k\).

3.2. Steady-state stage

As time passes, the thermal boundary-layer thickness \(\delta_T\) continues to grow until a balance between convection and conduction is reached. i.e.

\[
\frac{\Delta T}{t} \sim \kappa \frac{\Delta T}{\delta_T^2},
\]

\[
\Rightarrow \frac{u}{T} \sim \frac{1}{\delta_T}.
\]  
(8)

Using the velocity scale (7) and (8) we conclude that the growth of the thermal boundary layer along the inclined plate ends at a time of the order \(t_s\) given by

\[
\frac{\text{ARaPr}}{(1 + \text{Pr})(1 + A^2)^{1/2}} \left( \frac{t_s}{h^2/\kappa} \right) \left( \frac{k}{h} \right) \sim \frac{L}{t_s}
\]

\[
\Rightarrow t_s \sim \frac{(1 + \text{Pr})(1 + A^2)^{1/2}}{\text{ARa}^{1/2}\text{Pr}^{1/2}} \left( \frac{h^2}{\kappa} \right)\left( \frac{k}{h} \right).
\]  
(9)

The thickness of the thermal boundary layer along the plate at the steady-state time, \(t_s\) is

\[
\delta_T \sim \frac{h(1 + \text{Pr})^{1/4}(1 + A^2)^{1/4}}{A^{1/2}\text{Ra}^{1/4}\text{Pr}^{1/4}}.
\]  
(10)

At the time when the thermal boundary layer reaches the steady state, the \(u\) velocity scale is

\[
u_s \sim \frac{\text{Ra}^{1/2}\text{Pr}^{1/2}/k}{(1 + \text{Pr})^{1/2}(h/\kappa)}.
\]  
(11)

The thermal boundary-layer thickness at the steady state is shorter than the length of the plate if

\[
\delta_T < \frac{L}{t_s} < 1.
\]  
(12)

This is equivalent to having

\[
\frac{\text{Ra}^{1/2}\text{Pr}^{1/2}}{(1 + \text{Pr})^{1/2}} > \frac{k}{h}.
\]
Concurrently with the formation of a thermal boundary layer, a viscous boundary layer is developing. The thickness, \( \delta_v \), of this viscous layer is a direct result of a balance between the viscous and inertia terms in the momentum equation,

\[
\delta_v \sim (\nu t)^{1/2} \sim Pr^{1/3} \delta_t.
\]  

(13)

It is noted that for \( Pr < 1 \), the thickness of the viscous boundary layer is smaller than that of the thermal boundary layer. When the thermal layer has reached the steady state, the viscous layer has a thickness of the order

\[
\delta_v \sim \frac{h Pr^{1/4}(1 + Pr)^{1/4}(1 + A^2)^{1/4}}{A^{1/2} Ra^{1/4}}.
\]  

(14)

4. Possible flow regimes

Based on the above scale analysis we may define some regimes of the flow development depending on the Rayleigh number. Since the viscous boundary layer is smaller than the thermal boundary layer for \( Pr < 1 \), the viscous layer is always embedded in the thermal boundary layer. We may classify the flow development as follows:

(i) If \( Ra < A^2(1 + Pr^2)[Pr(1 + A^2)] \), the thermal boundary layer has grown to a thickness greater than the length of the heated plate at the steady state.

(ii) If \( Ra > A^2(1 + Pr^2)[Pr(1 + A^2)] \), the thermal boundary-layer thickness at the steady state is shorter than the length of the heated plate. That means the boundary-layer flow becomes steady before the thickness reaches a length scale equivalent to the length of the heated plate. For sufficiently high Rayleigh numbers the flow may become turbulent, which is beyond the scope of this thesis.

5. Scaling for ramp heating

The flow behavior adjacent to an inclined plate subject to a ramp temperature boundary condition is considered for \( Pr < 1 \). Scaling analysis for this boundary condition is still absent in the literature. However, Patterson et al. [14] have developed a scaling analysis for the ramp heating of a vertical flat plate with \( Pr > 1 \) and we follow that scaling here with \( Pr < 1 \). For this problem the physics of the system is the same as that for the sudden heating case which is depicted in Fig. 1. The plate CD = 1 is heated to \( T_h \) according to the following function.

\[
T_h = \begin{cases} 
T_c & \text{if } t \leq 0; \\
T_c + \Delta T(t / t_p) & \text{if } 0 < t < t_p; \\
T_c + \Delta T & \text{if } t \geq t_p;
\end{cases}
\]  

(15)

where \( \Delta T = T_h - T_c \) and \( t_p \) is the time duration of ramp heating.

Initially the flow is motionless and isothermal. As soon as the above temperature, \( T_h \), applied on the plate, a thermal boundary-layer develops adjacent to the heated inclined plate. The subsequent flow development is described in the following sections.

5.1. Early stage

The start-up stage is initially dominated by heat transfer via conduction through the hot plate, resulting in a thermal boundary layer of a thickness \( \delta_t \). As mentioned earlier for the case of the sudden heating boundary condition, initially the boundary-layer grows according to the scale \( x^{1/2} t^{1/2} \). Furthermore, for \( Pr \sim O(1) \), the unsteady and viscous terms together (i.e. \( (1 + Pr)u / t \)) balance the buoyancy term in the momentum equation;

\[
(1 + Pr) \frac{u}{t} \sim \frac{g \beta \sin \theta \Delta T}{t_p} \sim \frac{g \beta \sin \Delta T^2}{(1 + Pr)t_p} 
\]  

(16)

The above scale leads to the following velocity scale after substituting the Rayleigh number relation and the aspect ratio relation:

\[
\bar{u}_t \sim \frac{A Ra Pr}{(1 + Pr)(1 + A^2)} t_p^{1/2} \left( \frac{h}{2 \sqrt{k}} \right) \left( \frac{t}{t_p} \right)^{1/4} \left( \frac{k}{h} \right). 
\]  

(17)

This balance holds as long as \( t < t_s \).

5.2. Quasi-steady state time

The boundary-layer adjacent to the inclined plate continues to develop with the velocity scale defined in (17) and the thickness scale \( x^{1/2} t^{1/2} \) until the ramp finishes i.e. \( t < t_p \) or until a balance between convection and conduction is reached at time \( t_{sr} \) determined below:

\[
\bar{u}_t \sim \frac{A Ra Pr}{(1 + Pr)(1 + A^2)} t_p^{1/2} \left( \frac{h}{2 \sqrt{k}} \right) \left( \frac{t}{t_p} \right)^{1/4} \left( \frac{k}{h} \right). 
\]  

(18)

Using the velocity scales (17) and (18) we conclude that the growth of the thermal boundary layer along the inclined wall ends at time \( t_{sr} \) when

\[
\frac{A Ra Pr}{(1 + Pr)(1 + A^2)} \left( \frac{t}{t_p} \right)^{1/4} \sim \frac{h}{A t_{sr} \cos \theta} 
\]  

(19)

so long as \( t_{sr} < t_p \). This is the same as saying that

\[
\frac{t_p}{A^2} > \frac{(1 + Pr)(1 + A^2)^{1/2}}{A^{1/2} Ra^{1/2} \cos \theta} \left( \frac{h^2}{k} \right). 
\]  

(20)

The right-hand side of (20) represents the steady-state time scale for an instantaneous start-up function (refer to (9)). This means that if the ramp time is longer than the time it would take for the step function start-up to reach a steady-state boundary layer, then the boundary layer would have reached a convection–conduction balance before the ramp has finished.

The thickness of the thermal boundary layer along the plate at the time \( t_{sr} \) is

\[
\delta_{t_{sr}} \sim \frac{h(1 + Pr)^{1/6}(1 + A^2)^{1/6}}{A^{1/3}(Ra Pr)^{1/6} \left( \frac{h^2}{k} \right)}.
\]  

(21)

At the time when the plate boundary layer is steady, the \( u \) velocity scale is

\[
u_{t_{sr}} \sim \frac{(Ra Pr)^{1/3}(1 + A^2)^{1/6}}{A^{1/3}(1 + Pr)^{1/3} \left( \frac{h^2}{k} \right)} \left( \frac{k}{h} \right). 
\]  

(22)

On the other hand, if \( t_p < (1 + Pr)^{1/2}(1 + A^2)^{1/2}h^2/|A(Ra Pr)^{1/2}k| \), then \( t_{sr} > t_p \) and the thermal boundary layer has not finished growing when the ramp finishes. At the time when the ramp is finished (\( t = t_p \)) the unsteady velocity scale in the boundary layer is obtained from (17) as
\[ u \sim \frac{\text{ARaPr}}{(1 + \text{Pr})(1 + A^2)^{1/2}} \left( \frac{t_p}{R^2/k} \right)^{\left( \frac{1}{2} \right)} \]  

which is identical to the unsteady velocity scale (7) for the case of sudden heating boundary condition at the same time. The subsequent development of the boundary layer for \( t > t_p \) will follow the same thickness and velocity scales as those obtained for the sudden heating case until a steady state is reached. Hence there is no difference between the ramp and instantaneous start-up cases after the ramp is finished.

5.3. Quasi-steady stage

For the case for which the steady-state time is less than the ramp time, once the steady-state time \( t_{sr} \) is reached, the boundary layer stops growing according to \( \delta^2 \sim t_p \) which is only valid for conductive boundary layers. The thermal boundary layer is in a quasi-steady mode with convection balancing conduction. Further increase of the heat input simply accelerates the flow to maintain the proper thermal balance. For the ramp function start-up, this means that

\[ u \sim \frac{\Delta T}{t_p} \frac{t_p}{t_{sr}} \]

\[ \Rightarrow u \sim \frac{\kappa L}{\delta_T^2}. \]

At this time the unsteady term is not important because the ratio of the unsteady term to the viscous term is \( O(\delta_T^2/\nu T) \) and for large values of \( t, \delta_T^2/(\nu T \to 0) \). Therefore, the viscous term balances the buoyancy term. Hence,

\[ v \sim \frac{\delta_T^2}{\nu T} \frac{\Delta T}{t_p} \]

\[ \Rightarrow \delta_T \sim \frac{h(1 + A^2)^{1/4}}{A^{1/2}Ra^{1/4}} \frac{t_p}{t} \]

From (23) the velocity scale in the quasi-steady mode becomes

\[ u \sim Ra^{1/2}\kappa^2 \left( \frac{t}{t_p} \right)^{1/2}. \]

Notice that the boundary-layer thickness decreases beyond \( t_p \). This has to happen as the fluid is accelerating and is therefore more effective in convecting the heat away; the boundary layer has to contract so that conduction is increased to balance the increased convection.

In parallel with the formation of the thermal boundary layer for \( t < t_{sr} \), a viscous boundary layer also appears adjacent to the inclined plate with a balance between the viscous and inertia terms of the momentum equation, i.e.

\[ \delta_v \sim \nu^{1/2} t^{1/2} \sim Pr^{1/2} \delta_T \]

For \( Pr < 1 \), \( \delta_v \) is smaller than \( \delta_T \), implying that the viscous boundary layer is always embedded within the thermal boundary layer. When the thermal layer has reached the quasi-steady state (at \( t_{sr} \)), the viscous layer has a thickness of order

\[ \delta_{vt} \sim \frac{hPr^{1/3}(1 + Pr)^{1/6}(1 + A^2)^{1/6}}{A^{1/2}Ra^{1/6}} \left( \frac{t_p}{h^2/k} \right)^{1/6}. \]

However as the temperature on the plate continues to increase for \( t > t_{sr} \), the viscous boundary-layer thickness after the quasi-steady state is

### Table 1

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>Along x-axis</th>
<th>Plate GN</th>
<th>Both extended portions GN</th>
<th>Time step</th>
<th>EF</th>
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<th>EF</th>
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<td>1.008</td>
<td>0.0015</td>
<td></td>
</tr>
</tbody>
</table>

Note: GN is Grid Number, EF is expansion factor.

\[ \delta_v \sim \frac{hPr^{1/2}(1 + A^2)^{1/4}}{A^{1/2}Ra^{1/4}} \left( \frac{t_p}{t} \right)^{1/4} \]

Here it is also noted that the viscous boundary-layer thickness also decreases after the time \( t = t_{sr} \).

### Table 2

<table>
<thead>
<tr>
<th>Grid parameter for ( A = 0.05 ) and 1.0.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mesh size</td>
</tr>
<tr>
<td>-----------</td>
</tr>
<tr>
<td>320 × 200</td>
</tr>
<tr>
<td>330 × 150</td>
</tr>
<tr>
<td>440 × 200</td>
</tr>
<tr>
<td>660 × 300</td>
</tr>
</tbody>
</table>

### 6. Possible flow regimes

Similarly to the sudden heating case we may also classify the flow development under the ramp heating boundary condition into different flow regimes. It is found in the scaling analysis that the flow development depends on two time scales: the ramp time and the quasi-steady time. Based on these two time scales we may classify the flow development as follows:

(i) If \( Ra > (1 + \text{Pr})(1 + A^2)^{1/4}[(A^2/Pr)x^2] \), then the ramp time is longer than the quasi-steady time, and the flow becomes quasi-steady before the ramp is finished. Once conduction and convection are in balance, the thickness of the thermal boundary layer has reached a maximum.

If \( Ra > A^4(1 + Pr)x_{fs}[Prh^2(1 + A^2)^2] \), then the thermal boundary-layer thickness is shorter than the length of the heated plate. In this regime the flow is dominated by convection. The flow, however, may become turbulent for sufficiently high Rayleigh numbers. Turbulent analysis is out of the scope of this study.

(ii) If \( Ra < (1 + \text{Pr})(1 + A^2)^{1/4}[(A^2/Pr)x^2] \), then the ramp time is shorter than the steady/quasi-steady time and the boundary layer has not finished growing when the ramp is finished. What that means is that the boundary layer then grows as though the start-up was instantaneous and reaches a steady state at \( t_{sr} \). Therefore, there is no difference between the ramp and instantaneous start-up cases in this flow regime.

In the following sections, the above scaling relations are validated against the numerical simulation. However grid and time step dependence tests must first be performed to ensure the accuracy of the numerical results.
7. Numerical scheme and grid and time step dependence tests

The Fluent 6.3.26 software has been used to solve equations (1)–(4) along with the initial and boundary conditions using the SIMPLE scheme. The Finite Volume method has been chosen to discretize the governing equations with the QUICK scheme (see Leonard and Mokhtari [17]) approximating the advection term. The diffusion terms are discretized using central-differencing with second order accurate. A second order implicit time-marching scheme has also been used for the unsteady term. The details can be found in the user’s manual [18]. The associated Gambit 2.16 software is also used for the grid generation.

7.1. Grid generation

The resolution of the grid inside the computational domain plays an important role in the accuracy and the stability of
numerical simulations. In some regions of the domain a significant number of meshes are required in order to resolve the true physical flow features (e.g. boundary layers). A poorly distributed mesh in a critical region could result in false results. Unfortunately, it is often difficult to determine the locations of significance before the calculation is actually carried out. However we may use our previous knowledge to locate the regions of large flow gradients. Although an increase of the grid resolution generally increases the numerical accuracy, it also requires significant computing resources for both calculation and post-processing. Therefore, it is necessary to compromise between the numerical accuracy and computing efficiency when considering the numerical grid.

For natural convection adjacent to an inclined flat plate strong flows are present in the vicinity of the plate. Therefore, we need to distribute a non-uniform finer mesh near the plate when compared to other regions. We may use an expansion factor to distribute the non-uniform mesh. However, the expanding factor of grid is usually limited in order to ensure that the solution is not degraded. A factor of up to 10% may be used according to Patterson and Armfield [19].

The distribution of mesh tested is shown for three different aspect ratios in Tables 1 and 2. The grid distribution on the plate surface is uniform; however on the surface of the two extended ends of the plate an expansion factor has been used to form a non-uniform mesh. A non-uniform mesh has also been applied along the y-axis of the domain with finer mesh near the plate. A schematic of the grid distribution is shown in Fig. 2.

7.2. Test results

Grid and time step dependence tests have been conducted based on the numerical procedures described earlier for the highest Rayleigh number case for both thermal forcing conditions (that is, sudden heating and ramp heating). It is expected that the mesh selected for the highest Rayleigh number will also be applicable for all lower Rayleigh numbers.

The time histories of the calculated maximum velocity parallel to the sloping wall for different aspect ratios with four different meshes are plotted in Fig. 3 for the case of the sudden heating boundary condition. It is seen in the figure that all solutions indicate three stages of the flow development, an initial growth stage, a transitional stage and a steady-state stage. In the initial growth stage, the four solutions follow each other closely (except the solution with a coarse mesh 330 × 150, which deviates slightly from the other three meshes for A = 0.1 in Fig. 3(a)). The transitional stage is characterized by a single overshoot. The time to reach the steady state is around 0.8 s, 1.5 s and 6 s for A = 1.0, 0.5 and 0.1 respectively.

The maximum variation of the velocity between the coarsest and finest meshes for A = 0.1 is approximately 3.8%, and the maximum variation among the three fine meshes is only about 1.4%. The maximum variations of the velocity between the coarsest and finest meshes for A = 0.5 and 1.0 are 1.3% and 0.4% respectively. Therefore a fine mesh of 440 × 200 for A = 0.1 and a relatively coarse mesh 340 × 200 for A = 1.0 and 0.5 are adopted for the present simulations with a time step 0.002 s.

Mesh and time step dependence tests have also been conducted for the ramp heating boundary condition to ensure the accuracy of the numerical solutions. The same meshes as the sudden heating case have been considered here for three different aspect ratios. Fig. 4 shows the time series of the maximum velocity parallel to the inclined surface calculated along a line normal to the surface at the midpoint of the heated plate for three different aspect ratios for the ramp heating boundary condition for Ra = 3.0 × 10^7 and Pr = 0.72. The ramp time has been set to 20 s for all cases. As is mentioned in the scaling analysis, the ramp time may be longer or shorter than the steady-state time for the boundary layer. If the ramp time is longer than the steady-state time, then after the steady-state time the velocity continues to increase as the plate is

![Fig. 5. (a) Temperature contours, (b) streamlines and (c) temperature profiles along the line perpendicular to the plate at midpoint of the boundary-layer development for Ra = 10, Pr = 0.72 and A = 0.5 at t/τs = 2.4.](image-url)
still being heated. However, the growth rate of the velocity is reduced compared with that in the earlier phase. The two-stage growth of the velocity is clearly seen in the simulation results (see Fig. 4). It is seen in this figure that at about 12 s, 5.2 s and 4 s for $A = 0.1, 0.5$ and 1.0 respectively, the boundary layer becomes quasi steady. At $t = 20$ s the ramp finishes and the boundary layer becomes completely steady.

The maximum variation of the calculated maximum velocity between the coarsest and finest meshes for $A = 0.1, 0.5$ and 1.0 is 3.85%, 0.54% and 0.50% respectively. Therefore any of these meshes

Fig. 6. Temperature contours and streamlines of boundary-layer development for $Ra = 2.58 \times 10^7$, $Pr = 0.72$ and $A = 0.5$ at $t/t_s = 1.58$.

Fig. 7. (a) Temperature contours, (b) streamlines and (c) temperature profiles along the line perpendicular to the plate at midpoint of the boundary-layer development for $Ra = 5$, $Pr = 0.72$ and $A = 0.5$ at $t/t_s = 2.77$. 
is appropriate for this simulation, and the mesh size $440 \times 200$ is adopted for $A = 0.1$ and $340 \times 200$ is adopted for $A = 0.5$ and 1.0 for the following simulations with the time step size 0.002 s.

8. Flow development in different flow regimes for sudden heating

8.1. Conduction regime

The numerical results for a low Rayleigh number case are shown in Fig. 5 with $Pr = 0.72$, $Ra = 10$ and $A = 0.5$. The temperature contours and streamlines at $t/t_* = 2.4$ are plotted in Fig. 5(a) and (b), respectively. The heated portion of the inclined plate has been marked at the time of post-processing. This is also the case for subsequent Figs. 6–8. In this flow regime the thermal boundary layer eventually expands to the entire domain. Fig. 5(c) shows the temperature profile which has been extracted along a line perpendicular to the plate at the midpoint. The distance has been normalized by the length of the plate. It is seen in Fig. 5(c) that the thickness of the thermal boundary layer is larger than $y/L = 1$. Therefore, the flow is dominated by conduction in this regime.

8.2. Convection regime

The numerical results for a higher Rayleigh number with $Pr = 0.72$, $Ra = 2.58 \times 10^7$ and $A = 0.5$ at $t/t_* = 1.58$ are shown in Fig. 6. The temperature contours are presented in Fig. 6(a) and the streamlines are presented in Fig. 6(b). We notice that convection increases significantly in this regime as the Rayleigh number increases. The temperature contours are very much concentrated in the thin thermal boundary layer near the inclined plate as the result of strong convection. A temperature profile along a line perpendicular to the plate at the midpoint has been shown in Fig. 6(c). The thermal boundary-layer thickness can easily be deduced from this profile and is very small when compared to the length of the plate.
9. Flow development in different regimes for ramp heating

9.1. Ramp time shorter than steady-state time

Fig. 7 shows the temperature contours, the streamlines and a temperature profile for Pr = 0.72, Ra = 5 and A = 0.5 at t/t_{sr} = 2.77 where the length of the ramp time is t_p/t_{sr} = 0.042. Fig. 7(a) presents the temperature contours and Fig. 7(b) presents the corresponding streamlines. For this regime the steady-state time is larger than the ramp time. Therefore the flow behavior at the steady-state stage is identical to that for the sudden heating case. As soon as the heating starts, the thermal boundary-layer expands outwards from the heated plate and eventually arrives at the opposite wall of the rectangular domain as time passes. A temperature profile has been shown in Fig. 7(c) which is calculated along a line perpendicular to the plate at the midpoint. The distance is normalized by the length of the plate. It is seen that the thickness of the thermal boundary layer is larger than y/L = 1. Therefore, the flow is dominated by conduction in this regime.

9.2. Ramp time longer than steady time

In this regime, the flow becomes quasi-steady state before the ramp is finished. It is shown in the scaling development section that the boundary-layer thickness decreases beyond the quasi-steady time, t_{sr}. A representative Rayleigh number of Ra = 7.63 \times 10^5 has been chosen to demonstrate the flow features in this flow regime. The temperature contours and the streamlines are shown in Fig. 8 at different times of the boundary-layer development for an aspect ratio A = 0.5.

We notice in Fig. 8 that the boundary-layer develops adjacent to the plate and moves upwards. The ramp time, selected for this problem, is t_p/t_{sr} = 4.928. The isotherms and streamlines in Fig. 8(a) are at t/t_{sr} = 0.49, that is, before the flow becomes quasi-steady; those in Fig. 8(b) are at t/t_{sr} = 2.464 when the flow is in quasi-steady mode; and those in Fig. 8(c) are at the time when the ramp just finishes (t/t_{sr} = 4.928) and the flow is in a transitional stage from the quasi steady to the final steady state. We see from the start-up of the flow development to the steady state, the boundary layer is not affected significantly by the adiabatic walls which are artificial boundaries for forming a closed computational domain.

Fig. 9 shows the temperature profiles at two different times, t/t_{sr} = 2.464 and 4.924 respectively. At t/t_{sr} = 2.464 the flow just becomes quasi-steady and at t/t_{sr} = 4.924 the ramp time finishes.
It is seen in the temperature profiles that the thickness of the thermal boundary layer is smaller at the time 4.924 than that at 2.464. This supports the scaling relation (25) that the thermal boundary-layer contracts beyond the quasi-steady time $t_{ss}$.

10. Validation of selected scales

10.1. Scaling for sudden heating

The unsteady velocity scale (7), steady-state time scale (9), steady-state thermal layer thickness scale (10) and steady-state velocity scale (11) of the boundary-layer development for the case of sudden heating can be re-written in non-dimensional forms as

$$\frac{(1 + Pr) \frac{u}{(v/h)}}{ARaPr} \sim \frac{t}{h^2/k}$$

(30)

$$\frac{t_s}{R^2/k} \sim \frac{(1 + Pr) \frac{1}{1/4} (1 + A^2)^{1/4}}{ARa^{1/2}Pr^{1/4}}.$$  

(31)

$$\frac{\delta_T}{h} \sim (1 + Pr) \frac{1/4}{A^{1/2}Ra^{1/2}Pr^{1/4}}.$$  

(32)

$$\frac{u_s}{1/2} \sim \frac{Ra^{1/2}Pr^{1/2}}{\kappa/h} \frac{1}{1 + Pr^{1/2}}.$$  

(33)

In Table 3, Runs 1–5 with $Ra = 3.00 \times 10^7, 6.11 \times 10^6, 2.58 \times 10^6, 5.17 \times 10^5$ and $2.58 \times 10^5$ while keeping $A = 0.5$ and $Pr = 0.72$ unchanged have been carried out to show the dependence of the scaling relations on the Rayleigh number $Ra$; Runs 6–7 and 1 with $A = 1.0, 0.1$ and 0.5 respectively while keeping $Ra = 3.00 \times 10^7$ and $Pr = 0.72$ unchanged have been carried out to show the dependence on the slope of the inclination of the plate.

The velocity components and the temperature have been recorded at several locations along a line perpendicular to the plate at the midpoint to obtain the velocity and temperature profiles along that line. The maximum velocity parallel to the plate, $u_s$ has also been calculated from the velocity components and is used to verify the velocity scale relation.

The thermal boundary-layer thickness $\delta_T$ is determined as the perpendicular distance from the midpoint of the heated wall to the location where the temperature difference between the fluid in the thermal boundary layer and the ambient drops to 0.01($T_h - T_c$). The steady-state time, $t_s$ for the boundary-layer development to reach the steady state is determined as the moment when the first trough appears in the time history of the maximum parallel velocity, $u_s$ which is calculated along the line perpendicular to the plate at the midpoint (see Fig. 1).

The unsteady velocity scale (30) has been plotted in Fig. 10 for different parameters considered here, in which the $x$-axis is the normalized time and the $y$-axis includes the rest of the scale values. It is seen that all lines for different Rayleigh numbers and aspect ratios lie together initially on a straight line through the origin. This indicates that the scaling relation for the unsteady velocity is appropriate.

Numerical results supporting the scaling laws for the steady-state time, the steady-state thermal boundary-layer thickness and the steady-state velocity parallel to the plate, (31)–(33) respectively, are presented in Fig. 11. It is found in the figure that the

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**Fig. 12.** Velocity profiles (top) and temperature profiles (bottom) for 7 runs.
numerical results agree very well with these three scaling relations. For all the calculated cases, the numerical results fall approximately onto a straight line, which proves that the scaling relations (31)–(33) properly describe the thermal boundary layer at the steady state.

The velocity and temperature profiles at \( t/t_{sr} = 3.0 \) (when the flow is fully steady) are shown in Fig. 12 for different Rayleigh numbers and aspect ratios. Fig. 12(a) shows the raw data of the velocity along the line perpendicular to the plate at the midpoint. In Fig. 12(b), the velocity parallel to the plate has been normalized by its steady-state scale (11) and the distance normalized by its viscous boundary-layer thickness scale (14). Raw data of the temperature profiles is depicted in Fig. 12(c) at the same time for various Rayleigh numbers and aspect ratios. In Fig. 12(d), the temperature has been normalized by the maximum temperature difference \( \Delta T \) and the distance normalized by the steady-state thermal boundary-layer thickness (10).

The scaling relations for the steady-state velocity (11) and viscous boundary-layer thickness (14) are seen to work well for the velocity profiles. All profiles collapsing almost onto a single curve (see Fig. 12(b)). The scaling relation for the thermal boundary-layer thickness (10) also works very well as all temperature profiles for different parameters fall together. Therefore the scaling, derived from the sudden heating boundary condition, has been verified by the numerical simulation.

### 10.2. Scaling for ramp heating

A total of nine simulations have been performed to verify the scaling relations derived from the ramp heating boundary condition. Table 4 shows the details of the flow parameters considered for this study. Here, Runs 1–7 with the same aspect ratio \( A = 0.5 \) and Prandtl number \( \text{Pr} = 0.72 \) but different Rayleigh numbers have been carried out to show the dependence of the scaling relations on the Rayleigh number \( Ra \); and Runs 8–9 and 1 with \( A = 0.1, 1.0 \) and 0.5 respectively while keeping \( \text{Pr} = 0.72 \) and \( Ra = 3.0 \times 10^6 \) unchanged have been carried out to show the dependence of the scaling relations on the aspect ratio \( A \).

For this problem, the velocity parallel to the plate and the temperature have also been recorded at several locations along a line perpendicular to the plate at the midpoint to obtain the velocity and temperature profiles. Moreover, the maximum velocity parallel to the plate has been calculated as the characteristic velocity \( (u_{sr}) \) of the boundary layer, which is used to verify the velocity scale relation.

Fig. 13(a) shows the time series of the maximum velocity parallel to the inclined plate at the midpoint of the plate, where both the time and velocity are normalized with respect to their respective steady-state scaling values. It is clear that initially all lines collapse together; at about \( t/t_{sr} = 2.2 \) all curves bend together, indicating that at this time the flow reaches its quasi-steady mode. After that quasi-steady state, all curves continue to follow the same trend until the ramp is finished at respective times. This confirms the scaling relations (19) and (22).

To verify the scaling relation (25), \( uh/\sqrt{Ra} \) has been plotted against \( (t/t_{sr})^{1/2} \) for different Rayleigh numbers and aspect ratios with \( \text{Pr} = 0.72 \) in Fig. 13(b). This scaling is valid for \( t > t_{sr} \). It is seen that all lines after the quasi-steady state time fall approximately onto a single line. However, those cases for which the quasi-steady time and the ramp time are very close deviate a little from others. It is seen that after \( t/t_{sr} = 1.0 \), when ramp time finishes, all lines for different parameters lie together and form a horizontal line which confirms the scaling relation (26). A

Fig. 14 shows the velocity and temperature profiles for different Rayleigh numbers and aspect ratios along the line perpendicular to the plate at the midpoint at time \( t/t_{sr} = 2.6 \), when the flow becomes quasi steady. Raw velocity profiles for different aspect ratios and Rayleigh numbers have been shown in Fig. 14(a). In Fig. 14(b), the velocity is normalized with respect to the quasi-steady state velocity scale (22) and the position is normalized with respect to the viscous boundary-layer thickness scale (28). In Fig. 14(c) the raw temperature has been plotted against the normalized position with respect to the quasi-steady state thermal layer thickness. The temperature on the inclined plate does not reach the maximum temperature at this time as the ramp has not yet finished. Moreover, the instantaneous temperature differences are not the same at this time for different Rayleigh numbers and aspect ratios. The position of the temperature profile in Fig. 14(d) is also normalized by the quasi-steady state thermal boundary-layer thickness (21). However, the temperature is normalized by the instantaneous temperature difference (\( \Delta T_{inst} \)).

![Fig. 13](image.png) (a) Normalized velocity plotted against normalized time; (b) \( uh/\sqrt{Ra} \) plotted against \( (t/t_{sr})^{1/2} \).

### Table 4

<table>
<thead>
<tr>
<th>Runs</th>
<th>A</th>
<th>Ra</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>3.00 × 10^6</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>7.63 × 10^6</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>6.11 × 10^6</td>
</tr>
<tr>
<td>4</td>
<td>0.5</td>
<td>3.00 × 10^6</td>
</tr>
<tr>
<td>5</td>
<td>0.5</td>
<td>1.55 × 10^6</td>
</tr>
<tr>
<td>6</td>
<td>0.5</td>
<td>1.30 × 10^6</td>
</tr>
<tr>
<td>7</td>
<td>0.5</td>
<td>1.04 × 10^6</td>
</tr>
<tr>
<td>8</td>
<td>1.0</td>
<td>3.00 × 10^6</td>
</tr>
<tr>
<td>9</td>
<td>0.1</td>
<td>3.00 × 10^7</td>
</tr>
</tbody>
</table>
It is seen in Fig. 14(b) that all velocity profiles for different Ra and A fall into a single curve. Therefore, the scaling relations (22) and (28) are appropriate representations of the velocity and thickness respectively of the boundary layer. The same scenario can be seen in the temperature profiles in Fig. 14(d). All profiles fall onto a single line, confirming the scaling of the thermal boundary-layer thickness (21).

11. Conclusions

Natural convection adjacent to a heated inclined flat plate is examined by scaling analysis and verified by numerical simulations for air (Pr = 0.72). It is found that the flow is mainly dominated by three distinct stages for the sudden heating boundary condition, i.e. the start-up stage, the transitional stage and the steady-state stage.

The scaling relations are formed based on the established characteristic flow parameters of the maximum velocity inside the boundary layer ($u_s$), the time for the boundary layer to reach the steady state ($t_s$) and the thermal ($\Delta T$) and viscous ($\Delta n$) boundary-layer thickness. Through comparisons of those scaling assumptions with the numerical simulations, it is found that the scaling results agree very well with the numerical simulations. Hence the numerical results have confirmed the scaling relations which characterize the transient flow development.

References


